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# Approximation by complex Favard-Szász-Mirakjan-Stancu operators in compact disks

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**Abstract**

**Purpose:** The purpose of the present paper is to study the Stancu-type generalization of complex Favard-Szász-Mirakjan operators and establish some approximation results in complex domain.

**Methods:** It is observed that the complex Favard-Szász-Mirakjan-Stancu operators can be written in the form of divided differences. Thus, it is possible to study such operators in complex domain. We use analytical method to obtain our results.

**Results:** We have estimated the order of simultaneous approximation, Voronovskaja-type results with quantitative estimates for the complex Favard-Szász-Mirakjan-Stancu operators attached to analytic functions on compact disks. Also, some estimates on the rate of convergence are given.

**Conclusions:** The results proposed here are new and have better rate of convergence.

**Keywords:** Complex Favard-Szász-Mirakjan-Stancu operators, Voronovkaja-type result, Exact order of approximation in compact disks, Simultaneous approximation

**Mathematical Classification Subject:** 30E10; 41A25; 41A28

## Introduction

The Favard-Szász-Mirakjan operators are important and have been studied intensively, in connection with different branches of analysis, such as numerical analysis, approximation theory statistics etc. For a real function $f$ of real variable $f:[0,\infty)\to\mathbb{R}$, the Favard-Szász-Mirakjan operators are defined as follows:

$$S_n(f,x)=e^{-nx}\sum_{\nu=0}^{\infty}\frac{(nz)^{\nu}}{\nu!}f\left(\frac{\nu}{n}\right),\ x\in[0,\infty),$$

where the convergence of $S_n(f,x)\to f(x)$ under the exponential growth condition on $f$ that is $|f(x)|\le Ce^{Bx}$ for all $x\in[0,+\infty)$, with $C,B>0$ was established in [1]. The actual construction of the Szász-Mirakyan operators and its various modifications require estimations of infinite series which, in a certain sense, restrict their usefulness from the computational point of view. Recently Walczak [2] proposed and studied the Szász-Mirakyan operators by considering a finite sum.

Recently, Gal in his famous book [3] estimated Voronovskaja-type results with quantitative estimates for several complex operators. He also established exact order of simultaneous approximation by such complex operators. The Durrmeyer-type operators in complex domain were recently established in [4-12], etc. In [13], a Stancu-type generalization of the complex Durrmeyer operator was treated.

Also, Gal et. al [14] introduced complex Baskakov-Stancu operators and studied exact quantitative estimates and quantitative Voronovskaja-type results for these operators. Motivated by the recent study on this important topic, here, we deal with the following complex form for the Favard-Szász-Mirakjan-Stancu operator

$$S_n^{\alpha,\beta}(f,z)=\sum_{\nu=0}^{\infty}\left[\frac{\alpha}{n+\beta},\frac{\alpha+1}{n+\beta},\dots,\frac{\alpha+\nu}{n+\beta};f\right]z^{\nu},$$

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which for $\alpha = 0 = \beta$ was studied in [3]. Here, $[x_0, x_1, ..., x_m; f]$ denotes the divided difference of the function $f$ on the knots $x_0, x_1, ..., x_m$. We may note here that such a formula was first established by Lupas [15] for the special case $\alpha = \beta = 0$ and for functions of real variable. This formula holds for complex setting too, since only algebraic calculations were used in [15].

The aim of the present paper is to study the rate of approximation of analytic functions without exponential growth conditions and the Voronovskaja-type result for the Favard-Szász-Mirakjan-Stancu operator $S_n^{\alpha,\beta}(f,z)$. Also, the exact order of approximation by this operator is obtained.

## Methods

The principal methods used in the present work involve application of the theory of functions to analyze and study the order of simultaneous approximation, Voronovskaja-type results with quantitative estimates for the complex Favard-Szász-Mirakjan-Stancu operators attached to analytic functions on compact disks.

## Results and discussion

In the sequel, we need the following auxiliary results:

**Lemma 1.** *For all $n, k \in \mathbb{N} \cup \{0\}$, $0 \le \alpha \le \beta$, $z \in \mathbb{C}$, let us define*

$$S_n^{\alpha,\beta}(e_k, z) = \sum_{\nu=0}^{\infty} \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, ..., \frac{\alpha+\nu}{n+\beta}; e_k\right] z^\nu,$$

*where $e_k(z) = z^k$. Then, $S_n^{\alpha,\beta}(e_0, z) = 1$, and we have the following recurrence relation*

$$S_n^{\alpha,\beta}(e_{k+1}, z) = \frac{z}{n+\beta}(S_n^{\alpha,\beta}(e_k, z))' + \frac{nz+\alpha}{n+\beta} S_n^{\alpha,\beta}(e_k, z). \tag{1}$$

*Consequently,*

$$S_n^{\alpha,\beta}(e_1, z) = \frac{nz+\alpha}{n+\beta}, \quad S_n^{\alpha,\beta}(e_2, z) = \frac{n^2z^2}{(n+\beta)^2} + \frac{nz(1+2\alpha)}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}. \tag{2}$$

*Proof.* First, note that we have

$$S_n^{\alpha,\beta}(e_k, z) = \sum_{\nu=0}^{k} \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, ..., \frac{\alpha+\nu}{n+\beta}; e_k\right] z^\nu.$$

Simple calculation shows that the recurrence relation in the statement is equivalent to

$$\begin{aligned}\left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, ..., \frac{\alpha+\nu}{n+\beta}; e_{k+1}\right] &= \frac{\alpha+\nu}{n+\beta} \cdot \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, ..., \frac{\alpha+\nu}{n+\beta}; e_k\right] \\ &\quad + \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, ..., \frac{\alpha+\nu-1}{n+\beta}; e_k\right],\end{aligned} \tag{3}$$

which is an immediate consequence of the well-known relation (see [16], Exercise 4.9)

$$[x_0, x_1, ..., x_m; f \cdot g] = \sum_{i=0}^{m} [x_0, x_1, ..., x_i; f] \cdot [x_i, ..., x_m; g],$$

by taking $m = \nu$, $f = e_k$, $g = e_1$ and $x_i = \frac{\alpha+i}{n+\beta}$. □

**Lemma 2.** *Let $\alpha$, $\beta$ be satisfying $0 \le \alpha \le \beta$. Denoting $e_j(z) = z^j$ and $S_n^{0,0}(e_j)$ by $S_n(e_j)$ for all $n, k \in \mathbb{N} \cup \{0\}$, the following recursive relation for the images of the monomials $e_k$ under $S_n^{\alpha,\beta}$ in terms of $S_n(e_j)$, $j = 0, 1, 2..., k$,*

$$S_n^{\alpha,\beta}(e_k, z) = \sum_{j=0}^{k} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} S_n(e_j, z).$$

**Lemma 3.** *For all $n, k \in \mathbb{N} \cup \{0\}$, $0 \le \alpha \le \beta$ and $|z| \le r$, $r \ge 1$, we have*

$$\left|S_n^{\alpha,\beta}(e_k, z)\right| \le (2r)^k.$$

*Proof.* Denoting $e_k(z) = z^k$ for any $k \in \mathbb{N}$, we have

$$S_n^{\alpha,\beta}(e_k, z) = \sum_{\nu=0}^{k} \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, ..., \frac{\alpha+\nu}{n+\beta}; e_k\right] z^\nu.$$

Using the mean value theorem in complex analysis, we get

$$\begin{aligned}\left|S_n^{\alpha,\beta}(e_k, z)\right| &\le \sum_{\nu=1}^{k} \frac{k(k-1)...(k-\nu+1)}{\nu!} r^{k-\nu} r^\nu \\ &\le r^k \sum_{\nu=1}^{k} \frac{k(k-1)...(k-\nu+1)}{\nu!} \\ &= r^k \sum_{\nu=1}^{k} \binom{k}{\nu} \le (2r)^k,\end{aligned}$$

which proves the lemma. □

### Main results

Our first main result is the following theorem for upper estimates:

**Theorem 1.** *For $2 < R < +\infty$, let $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

(a) *Suppose that $0 \le \alpha \le \beta$ and $1 \le r < \frac{R}{2}$ are arbitrarily fixed. Then, for all $|z| \le r$ and $n \in \mathbb{N}$, we have*

$$\left|S_n^{\alpha,\beta}(f,z) - f(z)\right| \le \frac{\alpha+\beta r}{n+\beta}\sum_{k=1}^{\infty}|c_k| r^{k-1} + \frac{A_r(f)}{n+\beta} + \frac{\alpha B_r(f)}{n+\beta} + \frac{\beta C_r(f)}{n+\beta}, \tag{4}$$

*where $\sum_{k=1}^{\infty}|c_k| r^{k-1} < +\infty$, $B_r(f) = \sum_{k=1}^{\infty}|c_k| k r^{k-1} < +\infty$, $C_r(f) = \sum_{k=1}^{\infty}|c_k| k r^{k} < +\infty$ and $A_r(f) = 2\sum_{k=1}^{\infty}|c_k|(k-1)(2r)^{k-1} < +\infty$.*

(b) *Suppose that $0 \le \alpha \le \beta$ and $1 \le r < r_1 < \frac{R}{2}$, then for all $|z| \le r$ and $n \in \mathbb{N}$, we have*

$$\left|[S_n^{\alpha,\beta}(f,z)]^{(p)} - f^{(p)}(z)\right| \le \frac{p!\, r_1}{(r_1-r)^{p+1}} \cdot \frac{M_{r_1}(f)}{n+\beta},$$

*where $M_{r_1}(f) = (\alpha+\beta r_1)\sum_{k=1}^{\infty}|c_k| \cdot r_1^{k-1} + A_{r_1}(f) + B_{r_1}(f) + C_{r_1}(f)$.*

*Proof.* (a) Using the recurrence (1) in Lemma 1, we get

$$\begin{aligned} S_n^{\alpha,\beta}(e_k,z) - z^k =& \frac{z}{n+\beta}(S_n^{\alpha,\beta}(e_{k-1},z))' \\ &+ \frac{nz+\alpha}{n+\beta}\left(S_n^{\alpha,\beta}(e_{k-1},z) - z^{k-1}\right) \\ &+ \frac{nz+\alpha}{n+\beta} z^{k-1} - z^k \end{aligned} \tag{5}$$

and

$$\begin{aligned} |S_n^{\alpha,\beta}(e_k,z) - z^k| \le& \frac{|z|}{n+\beta}|(S_n^{\alpha,\beta}(e_{k-1},z))'| \\ &+ \frac{n|z|+\alpha}{n+\beta}\left|S_n^{\alpha,\beta}(e_{k-1},z) - z^{k-1}\right| \\ &+ \frac{\alpha}{n+\beta}|z|^{k-1} + \frac{\beta}{n+\beta}|z|^k. \end{aligned} \tag{6}$$

Clearly, $S_n^{\alpha,\beta}(e_0,z) - e_0 = 0$ and

$$\left|S_n^{\alpha,\beta}(e_1,z) - e_1(z)\right| = \left|\frac{nz+\beta}{n+\beta} - z\right| = \left|\frac{\alpha-\beta z}{n+\beta}\right| \le \frac{\alpha+\beta r}{n+\beta}. \tag{7}$$

Using Lemma 3 and Bernstein's inequality for the polynomial $S_n^{\alpha,\beta}(e_{k-1},z)$ of degree $\le k-1$, we have

$$\begin{aligned} \left|(S_n^{\alpha,\beta}(e_{k-1},z))'\right| &\le \frac{k-1}{r}\max\{|S_n^{\alpha,\beta}(e_{k-1},z)| : |z| \le r\} \\ &\le \frac{k-1}{r}(2r)^{k-1} = 2(k-1)(2r)^{k-2}. \end{aligned}$$

Therefore, it follows

$$\left|S_n^{\alpha,\beta}(e_k,z) - z^k\right|$$

$$\le \frac{k-1}{n+\beta}(2r)^{k-1} + \frac{nr+\alpha}{n+\beta}\left|S_n^{\alpha,\beta}(e_{k-1},z) - z^{k-1}\right| + \frac{\alpha}{n+\beta}r^{k-1} + \frac{\beta}{n+\beta}r^k \tag{8}$$

$$\le r\left|S_n^{\alpha,\beta}(e_{k-1},z) - z^{k-1}\right| + \frac{k}{n+\beta}(2r)^{k-1} + \frac{\alpha}{n+\beta}r^{k-1} + \frac{\beta}{n+\beta}r^k. \tag{9}$$

Taking above $k = 2$, we obtain

$$\left|S_n^{\alpha,\beta}(e_2,z) - e_2(z)\right| \le r \cdot \frac{\alpha+\beta r}{n+\beta} + \frac{1}{n+\beta}(2r) + \frac{\alpha}{n+\beta}r^1 + \frac{\beta}{n+\beta}r^2. \tag{10}$$

Then, for $k = 3$, it follows

$$\begin{aligned} \left|S_n^{\alpha,\beta}(e_3,z) - e_3(z)\right| \le& r^2 \cdot \frac{\alpha+\beta r}{n+\beta} + \frac{1\cdot 2^1}{n+\beta}r^2 + \frac{\alpha}{n+\beta}r^2 \\ &+ \frac{\beta}{n+\beta}r^3 + \frac{2\cdot 2^2}{n+\beta}r^2 + \frac{\alpha}{n+\beta}r^2 \\ &+ \frac{\beta}{n+\beta}r^3 = r^2 \cdot \frac{\alpha+\beta r}{n+\beta} \\ &+ \frac{1}{n+\beta}[(1\cdot 2^1) + (2\cdot 2^2)]\, r^2 \\ &+ \frac{2\alpha}{n+\beta}r^2 + \frac{2\beta}{n+\beta}r^3. \end{aligned} \tag{11}$$

Reasoning by recurrence for any $k \ge 2$, we finally get

$$\begin{aligned} \left|S_n^{\alpha,\beta}(e_k,z) - e_k(z)\right| \le& r^{k-1} \cdot \frac{\alpha+\beta r}{n+\beta} + \frac{1}{n+\beta}\cdot\left[\sum_{j=1}^{k-1} j\cdot 2^j\right]\cdot r^{k-1} \\ &+ \frac{(k-1)\alpha}{n+\beta}r^{k-1} + \frac{(k-1)\beta}{n+\beta}r^k \\ \le& r^{k-1}\cdot\frac{\alpha+\beta r}{n+\beta} + \frac{2(k-1)}{n+\beta}\cdot(2r)^{k-1} \\ &+ \frac{k\alpha}{n+\beta}r^{k-1} + \frac{k\beta}{n+\beta}r^k. \end{aligned} \tag{12}$$

Since the formula $\sum_{j=1}^{k-1} j2^j = (k-2)2^k + 2$ can easily be proved by mathematical induction, clearly, this inequality is valid for $k = 1$ too.

Now, reasoning exactly as in the case of complex Favard-Szász-Mirakjan operators in Remark 2 in [3], we can write

$$S_n^{\alpha,\beta}(f,z)=\sum_{k=0}^{\infty}c_k S_n^{\alpha,\beta}(e_k,z),$$

which implies

$$\begin{aligned}\left|S_n^{\alpha,\beta}(f,z)-f(z)\right| &\le \sum_{k=1}^{\infty}|c_k|\cdot|S_n^{\alpha,\beta}(e_k,z)-z^k|\\ &\le \frac{\alpha+\beta r}{n+\beta}\sum_{k=1}^{\infty}|c_k|\cdot r^{k-1}\\ &\quad+\frac{2}{n+\beta}\sum_{k=1}^{\infty}|c_k|(k-1)\cdot(2r)^{k-1}\\ &\quad+\frac{\alpha}{n+\beta}\sum_{k=1}^{\infty}|c_k|kr^{k-1}+\frac{\beta}{n+\beta}\sum_{k=1}^{\infty}|c_k|kr^k\\ &=\frac{\alpha+\beta r}{n+\beta}\sum_{k=1}^{\infty}|c_k|\cdot r^{k-1}\\ &\quad+\frac{A_r(f)}{n+\beta}+\frac{\alpha B_r(f)}{n+\beta}+\frac{\beta C_r(f)}{n+\beta}.\end{aligned} \tag{13}$$

Note here that by the analyticity of $f$, we clearly get $\sum_{k=1}^{\infty}|c_k|r^{k-1}<+\infty$, $B_r(f)<+\infty$, $C_r(f)<+\infty$ and $A_r(f)=2\sum_{k=1}^{\infty}|c_k|\cdot(k-1)\cdot(2r)^{k-1}<+\infty$, which proves (a).

(b) Denoting by $\gamma$, the circle of radius $r_1>r$ with center 0. For any $|z|\le r$ and $\nu\in\gamma$, we have $|\nu-z|\ge r_1-r$ and by Cauchy's formula for all $|z|\le r$, it follows

$$\begin{aligned}\left|[S_n^{\alpha,\beta}(f,z)]^{(p)}-f^{(p)}(z)\right| &= \frac{p!}{2\pi}\left|\int_{\gamma}\frac{S_n^{\alpha,\beta}(f,z)-f(z)}{(\nu-z)^{p+1}}\right|\\ &\le \frac{p!\,r_1}{(r_1-r)^{p+1}}\\ &\quad\cdot\left[\frac{\alpha+\beta r_1}{n+\beta}\sum_{k=1}^{\infty}|c_k|\cdot r_1^{k-1}+\frac{A_{r_1}(f)}{n+\beta}\right.\\ &\quad\left.+\frac{\alpha B_{r_1}(f)}{n+\beta}+\frac{\beta C_{r_1}(f)}{n+\beta}\right],\end{aligned} \tag{14}$$

which proves (b) and the theorem. □

The next main result is a Voronovskaja-type asymptotic formula.

**Theorem 2.** *For $2<R<+\infty$, let $f:[R,+\infty)\cup\overline{\mathbb{D}}_R\to\mathbb{C}$ be bounded on $[0,+\infty)$ and analytic in $\mathbb{D}_R$, that is $f(z)=\sum_{k=0}^{\infty}c_kz^k$, for all $z\in\mathbb{D}_R$. Also, let $1\le r<\frac{R}{2}$ and $0\le\alpha\le\beta$. Then, for all $|z|\le r$ and $n\in\mathbb{N}$, we have the following Voronovskaja-type result*

$$\left|S_n^{\alpha,\beta}(f,z)-f(z)-\frac{\alpha-\beta z}{n+\beta}f'(z)-\frac{z}{2n}f''(z)\right|\le\frac{M_{1,r}(f)}{n^2}+\frac{\sum_{j=2}^{6}M_{j,r}(f)}{(n+\beta)^2}, \tag{15}$$

*where*

$$\begin{aligned}M_{1,r}(f)&=26\sum_{k=3}^{\infty}|c_k|(k-1)^2(k-2)(2r)^{k-2}<+\infty,\ M_{2,r}(f)\\ &=\left(\frac{\alpha^2}{2}+2\alpha\right)\cdot\sum_{k=2}^{\infty}|c_k|\cdot k(k-1)(2r)^{k-2}<+\infty,\\ M_{3,r}(f)&=\frac{\beta^2}{2}\sum_{k=2}^{\infty}|c_k|k(k-1)(2r)^k<+\infty,\ M_{4,r}(f)\\ &=\beta\sum_{k=2}^{\infty}|c_k|k(k-1)(2r)^{k-1}<+\infty,\\ M_{5,r}(f)&=\alpha\beta\sum_{k=0}^{\infty}|c_k|k(k-1)r^{k-1}<+\infty,\ M_{6,r}(f)\\ &=\beta^2\sum_{k=0}^{\infty}|c_k|k(k-1)r^k<+\infty.\end{aligned} \tag{16}$$

*Proof.* For all $z\in\mathbb{D}_R$, let us consider

$$\begin{aligned}&S_n^{\alpha,\beta}(f,z)-f(z)-\frac{\alpha-\beta z}{n+\beta}f'(z)-\frac{z}{2n}f''(z)\\ &=S_n(f,z)-f(z)-\frac{z}{2n}f''(z)+S_n^{\alpha,\beta}(f,z)-S_n(f,z)-\frac{\alpha-\beta z}{n+\beta}f'(z).\end{aligned} \tag{17}$$

Taking $f(z)=\sum_{k=0}^{\infty}c_kz^k$, we get

$$\begin{aligned}&S_n^{\alpha,\beta}(f,z)-f(z)-\frac{\alpha-\beta z}{n+\beta}f'(z)-\frac{z}{2n}f''(z)\\ &=\sum_{k=2}^{\infty}c_k\left(S_n(e_k,z)-z^k-\frac{z}{2n}k(k-1)z^{k-2}\right)\\ &+\sum_{k=2}^{\infty}c_k\left(S_n^{\alpha,\beta}(e_k,z)-S_n(e_k,z)-\frac{\alpha-\beta z}{n+\beta}kz^{k-1}\right).\end{aligned}$$

To estimate the first sum, we use the Voronovskaja-type result for the Favard-Szász-Mirakjan operators obtained in [3], Theorem 1.8.5.

$$\left|S_n(f,z)-f(z)-\frac{z}{2n}f''(z)\right|\le\frac{26}{n^2}\sum_{k=3}^{\infty}|c_k|(k-1)^2(k-2)(2r)^{k-2}. \tag{18}$$

Next, to estimate the second sum, using Lemma 2, we obtain

$$
\begin{aligned}
&S_n^{\alpha,\beta}(e_k,z)-S_n(e_k,z)-\tfrac{\alpha-\beta z}{n+\beta}kz^{k-1}\\
&=\sum_{j=0}^{k-1}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}S_n(e_j,z)+\left(\frac{n^k}{(n+\beta)^k}-1\right)\\
&\qquad S_n(e_k,z)-\frac{\alpha-\beta z}{n+\beta}kz^{k-1}\\
&=\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}S_n(e_j,z)+\frac{kn^{k-1}\alpha}{(n+\beta)^k}S_n(e_{k-1},z)\\
&\qquad-\sum_{j=0}^{k-1}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}S_n(e_k,z)\\
&\qquad-\frac{\alpha-\beta z}{n+\beta}kz^{k-1}\\
&=\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}S_n(e_j,z)+\frac{kn^{k-1}\alpha}{(n+\beta)^k}[S_n(e_{k-1},z)-z^{k-1}]\\
&\qquad-\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}S_n(e_k,z)\\
&\qquad-\frac{kn^{k-1}\beta}{(n+\beta)^k}[S_n(e_k,z)-z^k]\\
&\qquad+\left(\frac{n^{k-1}}{(n+\beta)^{k-1}}-1\right)\frac{k\alpha}{n+\beta}z^{k-1}\\
&\qquad+\left(1-\frac{n^{k-1}}{(n+\beta)^{k-1}}\right)\frac{k\beta}{n+\beta}z^k.
\end{aligned}
\tag{19}
$$

Now, using the inequalities

$$1-\frac{n^k}{(n+\beta)^k}\le\sum_{j=1}^{k}\left(1-\frac{n}{n+\beta}\right)=\frac{k\beta}{n+\beta}, \tag{20}$$

$$
\begin{aligned}
\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{n^j\alpha^{k-2-j}}{(n+\beta)^{k-2}}&=\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{n^j}{(n+\beta)^j}\\
&\quad\cdot\frac{\alpha^{k-2-j}}{(n+\beta)^{k-2-j}}=\left(\frac{n+\alpha}{n+\beta}\right)^{k-2}\le 1,
\end{aligned}
\tag{21}
$$

$$|S_n(e_j,z)|\le(2r)^k \text{ (see Lemma 3 for } \alpha=\beta=0),$$

$$|S_n(e_k,z)-z^k|\le\frac{k-1}{n}\cdot(2r)^{k-1},$$

for this last inequality take $\alpha=\beta=0$ in the proof of the above Theorem 1

and

$$
\begin{aligned}
\left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}S_n(e_j,z)\right|&\le\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}|S_n(e_j,z)|\\
&=\sum_{j=0}^{k-2}\frac{k(k-1)}{(k-j)(k-j-1)}\binom{k-2}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}|S_n(e_j,z)|\\
&\le\frac{k(k-1)}{2}\cdot\frac{\alpha^2}{(n+\beta)^2}(2r)^{k-2}\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{n^j\alpha^{k-2-j}}{(n+\beta)^{(k-2)}}\\
&\le\frac{k(k-1)}{2}\cdot\frac{\alpha^2}{(n+\beta)^2}(2r)^{k-2},
\end{aligned}
\tag{22}
$$

it follows

$$
\begin{aligned}
&\left|S_n^{\alpha,\beta}(e_k,z)-S_n(e_k,z)-\frac{\alpha-\beta z}{n+\beta}kz^{k-1}\right|\\
&\le\left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\alpha^{k-j}}{(n+\beta)^k}S_n(e_j,z)\right|\\
&\quad+\frac{kn^{k-1}\alpha}{(n+\beta)^k}\left|S_n(e_{k-1},z)-z^{k-1}\right|\\
&\quad+\left|\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}S_n(e_k,z)\right|\\
&\quad+\frac{kn^{k-1}\beta}{(n+\beta)^k}|S_n(e_k,z)-z^k|\\
&\quad+\left|\left(\frac{n^{k-1}}{(n+\beta)^{k-1}}-1\right)\right|\frac{k\alpha}{n+\beta}|z|^{k-1}\\
&\quad+\left|\left(1-\frac{n^{k-1}}{(n+\beta)^{k-1}}\right)\right|\frac{k\beta}{n+\beta}|z|^k\\
&\le\frac{k(k-1)\alpha^2}{2(n+\beta)^2}(2r)^{k-2}+\frac{kn^{k-1}\alpha}{(n+\beta)^k}\cdot\frac{(2r)^{k-2}(k-2)}{n}\\
&\quad+(2r)^k\cdot\sum_{j=0}^{k-2}\binom{k}{j}\frac{n^j\beta^{k-j}}{(n+\beta)^k}\\
&\quad+\frac{kn^{k-1}\beta}{(n+\beta)^k}\cdot\frac{(2r)^{k-1}(k-1)}{n}+\frac{k(k-1)\alpha\beta}{(n+\beta)^2}r^{k-1}\\
&\quad+\frac{k(k-1)\beta^2}{(n+\beta)^2}r^k,
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{k(k-1)\alpha^2}{2(n+\beta)^2}(2r)^{k-2} \\
&\quad + \frac{k(k-2)\alpha}{(n+\beta)^2}\cdot(2r)^{k-2} \\
&\quad + \frac{\beta^2 k(k-1)}{2(n+\beta)^2}(2r)^k \\
&\quad + \frac{k(k-1)\beta}{(n+\beta)^2}(2r)^{k-1} \\
&\quad + \frac{k(k-1)\alpha\beta}{(n+\beta)^2}r^{k-1} + \frac{k(k-1)\beta^2}{(n+\beta)^2}r^k \\
&\leq \frac{k(k-1)}{(n+\beta)^2}\left(\frac{\alpha^2}{2}+2\alpha\right)(2r)^{k-2} \\
&\quad + \frac{\beta^2 k(k-1)}{2(n+\beta)^2}(2r)^k + \frac{k(k-1)\beta}{(n+\beta)^2}(2r)^{k-1} \\
&\quad + \frac{k(k-1)\alpha\beta}{(n+\beta)^2}r^{k-1} + \frac{k(k-1)\beta^2}{(n+\beta)^2}r^k,
\end{aligned} \tag{23}$$

which immediately proves the theorem. □

Now, we will give the exact order of approximation by the operators $V_n^{\alpha,\beta}$.

**Theorem 3.** *For $2 < R < +\infty$, $1 \leq r < \frac{R}{2}$, let $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.*

*Let $0 < \alpha \leq \beta$ and suppose that $f$ is not a polynomial of degree $\leq 0$. Then, for all $n \in \mathbb{N}$ and $|z| \leq r$, we have*

$$\left|S_n^{\alpha,\beta}(f,z) - f(z)\right| \geq \frac{C_r(f)}{n},$$

*where the constant $C_r(f)$ depends only on $f$, $\alpha$, $\beta$ and $r$.*

*Proof.* For all $|z| \leq r$ and $n \in \mathbb{N}$, we can write

$$\begin{aligned}
S_n^{\alpha,\beta}(f,z) - f(z) &= \frac{1}{n}\Bigg[\frac{n}{n+\beta}(\alpha-\beta z)f'(z) + \frac{z}{2}f''(z) \\
&\quad + \frac{1}{n}\cdot n^2\Big(S_n^{\alpha,\beta}(f,z) - f(z) \\
&\quad - \frac{\alpha-\beta z}{n+\beta}f'(z) - \frac{z}{2n}f''(z)\Big)\Bigg] \\
&= \frac{1}{n}\Bigg[(\alpha-\beta z)f'(z) + \frac{z}{2}f''(z) \\
&\quad + \frac{1}{n}\cdot n^2\Big(S_n^{\alpha,\beta}(f)(z) - f(z) \\
&\quad - \frac{\alpha-\beta z}{n+\beta}f'(z) \\
&\quad - \frac{z}{2n}f''(z) - \frac{\beta(\alpha-\beta z)f'(z)}{n(n+\beta)}\Big)\Bigg].
\end{aligned} \tag{24}$$

Applying the inequality

$$\|F+G\| \geq |\,\|F\| - \|G\|\,| \geq \|F\| - \|G\|,$$

we obtain

$$\begin{aligned}
\|S_n^{\alpha,\beta}(f) - f\|_r &\geq \frac{1}{n}\Bigg[\left\|(\alpha-\beta e_1)f' + \frac{e_1}{2}f''\right\|_r \\
&\quad - \frac{1}{n}\cdot n^2\Big\|S_n^{\alpha,\beta}(f) - f - \frac{\alpha-\beta e_1}{n+\beta} \\
&\quad - \frac{e_1}{2n}f'' - \frac{\beta(\alpha-\beta e_1)f'}{n(n+\beta)}\Big\|_r\Bigg].
\end{aligned} \tag{25}$$

Since $f$ is not a polynomial of degree $\leq 0$ in $\mathbb{D}_R$, we get $\left\|(\alpha-\beta e_1)f' + \frac{e_1}{2}f''\right\|_r > 0$. Indeed, supposing the contrary, it follows that

$$(\alpha-\beta z)f'(z) + \frac{z}{2}f''(z) = 0, \text{ for all} z \in \overline{\mathbb{D}}_r.$$

Denoting $y(z) = f'(z)$, seeking $y(z)$ in the form $y(z) = \sum_{k=0}^{\infty} b_k z^k$ and replacing in the above differential equation, we easily get $b_k = 0$ for all $k = 0, 1, \ldots$, (we can make here similar reasonings with those in [17]; see also [3]). Thus, we get that $f(z)$ is a constant function, which is a contradiction.

Now, since by Theorem 2 it follows

$$\begin{aligned}
&n^2\left\|S_n^{\alpha,\beta}(f) - f - \frac{\alpha-\beta e_1}{n+\beta}f' - \frac{e_1}{2n}f'' - \frac{\beta(\alpha-\beta e_1)f'}{n(n+\beta)}\right\|_r \\
&\leq n^2\left\|S_n^{\alpha,\beta}(f) - f - \frac{\alpha-\beta e_1}{n+\beta}f' - \frac{e_1}{2n}f''\right\|_r + \|\beta(\alpha-\beta e_1)f'\|_r \\
&\leq \sum_{j=1}^{6} M_{j,r}(f) + \beta(\alpha+\beta r)\|f'\|_r,
\end{aligned} \tag{26}$$

there exists $n_1 > n_0$ (depending on $f$, $\alpha$, $\beta$ and $r$ only) such that for all $n \geq n_1$, we have

$$\begin{aligned}
&\left\|(\alpha-\beta e_1)f' + \frac{e_1}{2}f''\right\|_r \\
&\quad - \frac{1}{n}\cdot n^2\Big\|S_n^{\alpha,\beta}(f) - f - \frac{\alpha-\beta e_1}{n+\beta}f' - \frac{e_1}{2n}f'' \\
&\quad - \frac{\beta(\alpha-\beta e_1)f'}{n(n+\beta)}\Big\|_r \geq \frac{1}{2}\left\|(\alpha-\beta e_1)f' + \frac{e_1}{2}f''\right\|_r,
\end{aligned} \tag{27}$$

which implies that

$$\|S_n^{\alpha,\beta}(f) - f\|_r \geq \frac{1}{2n}\left\|(\alpha-\beta e_1)f' + \frac{e_1}{2}f''\right\|_r$$

for all $n \geq n_1$.

For $n \in \{n_0+1, \ldots, n_1\}$, we get $\|S_n^{\alpha,\beta}(f) - f\|_r \geq \frac{1}{n}A_r(f)$ with $A_r(f) = n\cdot\|S_n^{\alpha,\beta}(f) - f\|_r > 0$, which implies that

$\|S_n^{\alpha,\beta}(f) - f\|_r \geq \frac{C_r(f)}{n}$ for all $n \geq n_0$, with

$$C_r(f) = \min\left\{A_{r,n_0+1}(f), ...., A_{r,n_1}(f), \frac{1}{2}\left\|(\alpha - \beta e_1)f' + \frac{e_1}{2}f''\right\|_r\right\}, \tag{28}$$

which proves the theorem. □

**Remark 1.** *By Theorems 1 and 3, it easily follows that if f is not a constant function, then the exact order in the approximation by the complex Favard-Szász-Mirakjan-Stancu operator $S_n^{\alpha,\beta}$ is $\frac{1}{n}$.*

Concerning the simultaneous approximation, we present the following:

**Theorem 4.** *For $2 < R < +\infty$, $1 \leq r < \frac{R}{2}$, let $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \to \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in $\mathbb{D}_R$, that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$ and let $1 \leq r < r_1 < \frac{R}{2}$ and $p \in \mathbb{N}$ be fixed.*

*Let $0 < \alpha \leq \beta$ and suppose that f is not a polynomial of degree $\leq p-1$. Then, for all $n \in \mathbb{N}$ and $|z| \leq r$, we have*

$$\|[S_n^{\alpha,\beta}(f)]^{(p)} - f^{(p)}\|_r \sim \frac{1}{n},$$

*where the constants in the equivalence depend only on f (that is on M and A), α, β, p and r.*

*Proof.* Denoted by $\gamma$, the circle of radius $r_1$ with $r < r_1 < \min\{n_0/2, 1/A\}$ and center 0. Since for $|z| \leq r$ and $v \in \gamma$ we have $|v - z| \geq r_1 - r$, by the Cauchy's formula, for all $|z| \leq r$ and $n > n_0$, we obtain

$$\left|[S_n^{\alpha,\beta}(f,z)]^{(p)} - f^{(p)}(z)\right| = \frac{p!}{2\pi} \cdot \left|\int_\gamma \frac{S_n^{\alpha,\beta}(f,v) - f(v)}{(v-z)^{p+1}} dv\right| \leq \frac{C_{r,r_1,\alpha,\beta}}{n} \cdot \frac{p!}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r)^{p+1}}, \tag{29}$$

which proves one of the inequalities in the equivalence. Note here that $r_1$ depends in fact on $r$, $A$ and $n_0$; therefore, it depends in fact on $r$ and $f$.

To prove the converse inequality in the equivalence, we start from the relationship for $S_n^{\alpha,\beta}(f,v) - f(v)$ in (1) (in the proof of Theorem 3, with $v$ instead of $z$ there), replaced in the Cauchy's formula

$$[S_n^{\alpha,\beta}(f,z)]^{(p)} - f^{(p)}(z) = \frac{p!}{2\pi i} \cdot \int_\gamma \frac{S_n^{\alpha,\beta}(f,v) - f(v)}{(v-z)^{p+1}} dv.$$

By standard reasonings as those for the case of classical complex Favard-Szász-Mirakjan operator (see the proof of Theorem 1.8.6 in [3]), combined with those for the Bernstein-Stancu polynomials (see [17] or the proof of Theorem 1.6.5 in [3]), the present proof finally reduces to the proof of the fact that $\|[(\alpha - \beta z)f'(z) + \frac{z}{2}f''(z)]^{(p)}\|_r > 0$. However, this can be shown by following exactly the lines in [17]; see also [3]. As the reasonings are standard, we omit the details. □

## Conclusions

For the parameters $\alpha$ and $\beta$, our results have better rate convergence for the complex Favard-Szász-Mirakjan-Stancu operators which include $\alpha = 0 = \beta$, as special case. In special case, the Theorems 1, 2, 3 and 4 become the results in [7]; see also [3].

#### Competing interests
Both authors declare that they have no competing interests.

#### Authors' contributions
VG and DKV contributed equally to this work. Both authors read and approved the final manuscript.

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